

PARABOLIC OPTIMAL TRANSPORT EQUATIONS ON MANIFOLDS

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ABSTRACT. We study a parabolic equation for finding solutions to the optimal transport problem on compact Riemannian manifolds with general cost functions. We show that if the cost satisfies the strong MTW condition and the stay-away singularity property, then the solution to the parabolic flow with any appropriate initial condition exists for all time and it converges exponentially to the solution to the optimal transportation problem. Such results hold in particular, on the sphere for the distance squared cost of the round metric and for the far-field reflector antenna cost, among others.

1. INTRODUCTION

In this paper we study a fully nonlinear parabolic flow toward optimal transport maps between smooth densities on compact manifolds. Let M, \bar{M} be two n -dimensional closed (i.e. compact and without boundary) Riemannian manifolds equipped with the transportation cost $c : M \times \bar{M} \ni (x, \bar{x}) \mapsto c(x, \bar{x}) \in \mathbf{R} \cup \{+\infty\}$. For example, $M = \bar{M}$ and $c(x, \bar{x}) = \text{dist}(x, \bar{x})^2$ where dist denotes the Riemannian distance function of the given metric. As we see the presence of cut-locus for dist , the transportation cost may not be smooth, and we denote $\text{sing}(c) \subset M \times \bar{M}$ to be the set of points where c fails to be C^∞ . Let $\rho, \bar{\rho}$ be two smooth probability measures on M, \bar{M} , respectively. (We use $\rho(x), \bar{\rho}(\bar{x})$ to express the corresponding densities in local coordinates, i.e. $\rho = \rho(x)dx, \bar{\rho} = \bar{\rho}(\bar{x})d\bar{x}$.)

One seeks an optimal map $T : M \rightarrow \bar{M}$, which minimizes the cost functional

$$\int_M c(x, F(x)) \rho dx$$

among all maps F which satisfy $F_\# \rho = \bar{\rho}$, i.e., $\int_M f(F(x)) \bar{\rho}(x) dx = \int_{\bar{M}} f(\bar{x}) \bar{\rho}(\bar{x}) d\bar{x}$ for all $f \in C^\infty(\bar{M})$.

It is well known (c.f. [Br, GM, Mc, MTW]) that such an optimal map uniquely exists under some appropriate conditions on $c, \rho, \bar{\rho}$, and it always is associated with

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a potential function $u : M \rightarrow \mathbf{R}$, such that the optimal map T (if smooth) satisfies

$$(1.1) \quad -D_x c(x, T(x)) = D_x u(x).$$

Here, D_x denotes the derivative in the first variable $x \in M$. Note that the condition $T_{\#}\rho = \bar{\rho}$ forces the potential u (if smooth) to satisfy a Monge-Ampère type equation

$$(1.2) \quad \det(u_{ij}(x) + c_{ij}(x, T(x))) = \frac{\rho(x)}{\bar{\rho}(T(x))} |\det c_{\bar{i}\bar{j}}(x, T(x))|$$

where the subindices i, j, \dots denote the derivative (in local coordinates) in the first variable x while the barred subindices \bar{i}, \bar{j}, \dots denote the one for the second variable \bar{x} .

The seminal work of Ma, Trudinger and Wang [MTW, TW1] (which follows that of Delanoë [D], Caffarelli [Ca1, Ca2] and Urbas [U]) studies regularity of optimal maps for general cost functions c by considering regularity of solutions to the fully nonlinear (degenerate) elliptic equation (1.2). Importantly, they have identified a structure condition on c , now widely called the MTW condition (see 2.1 for more details), which later is shown by Loeper [Lo1] to be a necessary condition for the regularity of solutions to (1.2). This then is followed by many works including those of (in alphabetical order) Delanoë, Figalli, Ge, Kim, Liu, Loeper, McCann, Rifford, Trudinger, Villani, Wang and Warren [Lo1, Lo2, TW2, KM1, Km, KM2, DG, LV, V, FR, LTW, FKM1, KMW, FKM2, FKM3, FRV], among others, which study both regularity of optimal maps and geometric issues related to the MTW condition.

With this in mind, in the present paper we study the following fully nonlinear parabolic equation for $u : M \times [0, \infty) \rightarrow \mathbf{R}$,

$$(1.3) \quad \frac{\partial u}{\partial t} = \ln \det(u_{ij} + c_{ij}(x, T(x))) - \ln \rho(x) + \ln \bar{\rho}(T(x)) - \ln \det |c_{\bar{i}\bar{j}}(x, T(x))|.$$

Here, to define T uniquely by u through (1.1) and to make the logarithm make sense, we assume (as in [MTW]) throughout the present paper, the following conditions for the cost function:

$$(1.4) \quad \text{the map } \bar{x} \mapsto D_x c(x, \bar{x}) \in T_x^* M \text{ is injective for } (x, \bar{x}) \in M \times \bar{M} \setminus \text{sing}(c);$$

$$(1.5) \quad \det c_{\bar{i}\bar{j}}(x, \bar{x}) \neq 0 \text{ on } M \times \bar{M} \setminus \text{sing}(c).$$

Notice that the right hand side in (1.3) is a coordinate invariant quantity.

The main result of the present paper is the following

Theorem 1.1 (Parabolic flow toward optimal transport on manifolds). *Let M, \bar{M} be two n -dimensional compact Riemannian manifolds without boundary, equipped with a cost function $c : M \times \bar{M} \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfying (1.4) and (1.5). Assume that c is locally semi-concave on the domain where its value is finite. Let $\rho, \bar{\rho}$ be smooth, positive, probability densities on M, \bar{M} , respectively. Assume further c satisfies (i) MTW(δ) condition for some $\delta > 0$, and (ii) stay-away-from-singularity property (see Sections 2.1 and 2.2 for definitions). Let $u_0 : M \rightarrow \mathbf{R}$ be a C^2 locally strictly c -convex function: see (2.5) for definition. Then, there exists a unique smooth solution $u : M \times [0, \infty) \rightarrow \mathbf{R}$, of (1.3) with $u(\cdot, 0) \equiv u_0$. Moreover, the*

derivatives of u (i.e. $\|u\|_{C^m}$ for each $m \in \mathbb{N}$) are bounded uniformly in time, and as $t \rightarrow \infty$, $u(\cdot, t)$ converges exponentially to a solution of the equation (1.2), and this solution to (1.2) defines the unique solution to the optimal transportation problem.

Remark 1. In fact, as a corollary to Theorem 1.1, one can show that C^2 locally strictly c -convex functions are indeed globally c -convex in the situation of Theorem 1.1. See Corollary 7.1 (see Section 2.2 for definitions).

The main consequences of Theorem 1.1 are in the following

Corollary 1.2. *The same existence and exponential convergence result for the solution to (1.3) as in Theorem 1.1 holds for the following cases of $M = \bar{M}$, where the condition $\text{MTW}(\delta)$, $\delta > 0$, and the stay-away-from-singularity property as well as regularity of solutions to the elliptic equation (1.2) are shown in the papers cited correspondingly :*

- (1) $c = \text{dist}^2/2$ on the round sphere (see [DL, Lo2] for elliptic case);
- (2) $c = \text{dist}^2/2$ on small C^4 perturbations of the round sphere, but in a restricted sense that λ_1, λ_2 in (2.4) are restricted by the size of perturbation (see [DG, FRV1] for elliptic case);
- (3) $c = \text{dist}^2/2$ on the covering quotient S^n/Γ , Γ discrete group, e.g. \mathbf{RP}^n , (see [DG, KM2] for elliptic case), and their C^4 perturbations (see [LV] for elliptic case).
- (4) $c = \text{dist}^2/2$ on the complex project space \mathbf{CP}^n and the quaternionic n -space \mathbf{HP}^n with the metric induced from the round sphere by the Hopf fibrations $S^{2n+1} \rightarrow \mathbf{CP}^n$ and $S^{4n+3} \rightarrow \mathbf{HP}^n$ (see [KM2] for elliptic case).
- (5) the far-field reflector antenna cost $c(x, \bar{x}) = -\log|x - \bar{x}|$ on the imbedded round sphere $S^n \subset \mathbf{R}^{n+1}$ (Schnürer [Sch] gave a proof in parabolic case, see [Lo2] for elliptic case.)

Remark 2. In all of the above cases one can take $u_0 \equiv 0$ as an initial condition.

Remark 3. Recently, in [FKM3] regularity of solutions to the elliptic equation (1.2) is shown for $c = \text{dist}^2/2$ on the multiple products $S_{r_1}^{n_1} \times \cdots \times S_{r_k}^{n_k}$ of round spheres of arbitrary dimension n_i and size r_i , by showing the stay-away-from-singularity property and using the results in [FKM1, LTW]. However, we do not have the corresponding parabolic version yet.

The proof of Theorem 1.1 is given in the rest of the paper; especially, see Sections 4 and 7. We use the tensor maximum principle method for obtaining the second derivative estimates, which is essentially the calculation of Ma, Trudinger and Wang [MTW], given the stay away assumption. We use the parabolic Krylov-Safonov theory to obtain $C^{2,\alpha}$ estimates from $C^{1,1}$ estimates, although we note that a long and straightforward parabolic adaptation of the Calabi computation in [CNS] would furnish the $C^{1,1}$ to $C^{2,1}$ estimates, if one were interested in a “bare-hands” proof. Higher estimates follow by the Schauder theory. In order to obtain the exponential convergence, instead of relying on the quite general Krylov-Safonov

Harnack estimates, we directly prove Li-Yau type Harnack estimates. Also, a simple topological argument is used to show that the resulting limit function at $t = \infty$ is indeed the solution to the optimal transport problem.

The proof of Li-Yau type estimates illustrates the natural relevance of the Riemannian submanifold geometry that is first seen in [KM1], further refined in [KMW] and then again in [Wr]. In this paper, we see that the quantity θ , defined to be the right-hand side of (1.3), which measures how far the map at a given time is from being volume preserving, satisfies a heat-like equation, which makes it vulnerable to the Li-Yau approach. For nonlinear equations, the idea of using the linearized operator to define a Laplacian with respect to a metric goes back to Calabi [Cb], and also arises naturally when studying minimal submanifold equations. In our case, there is an interesting conformal factor, which forces us to perform a slight workaround in two dimensions. See Section 5.

Having practical applications in mind, the exponential convergence in Theorem 1.1 can be of independent interest since one can view our parabolic flow as an algorithm to construct the solution to optimal transport problem. Indeed, Schnurer [Sch] applied a parabolic flow to construct reflector antennas for given light sources: this practical problem in geometric optics has been known to be an optimal transport problem as found by X.-J. Wang [Wg]. Recently, Kitagawa [Kt] has a result similar to ours, which deals with oblique boundary value problem on domains in \mathbf{R}^n , for cost functions satisfying the MTW(0) condition (see Section 2.1).

The paper is organized as follows. First, after a few preliminaries in Section 2, we obtain in Section 3 the stay-away-from-singularity property along the flow which is then used together with the maximum principle argument of Ma, Trudinger, and Wang [MTW] applied to the parabolic setting, to get uniform second derivative estimates along the flow. In Section 4 we use these estimates to obtain the long time existence and uniform derivative bounds of the solution. Following some remarks about the linearized operator in Section 5, we prove the Harnack inequality in Section 6. Finally, in Section 7, the exponential convergence to the stationary solution at infinity is obtained, and this stationary solution is shown to be the solution to the optimal transport problem, thus finishing the proof of Theorem 1.1. As a corollary to Theorem 1.1, it is shown in Corollary 7.1 that C^2 locally c -convex functions are globally c -convex.

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2. PRELIMINARIES

2.1. Ma-Trudinger-Wang curvature. We first explain the MTW curvature that is first introduced by Ma, Trudinger and Wang [MTW] as a quantity which can be used to guarantee interior regularity for solutions of the elliptic equation (1.2). This notion is further investigated by Loeper [Lo1] and then by Kim and McCann [KM1] (see also [Km, KM2, LV, V, FR, FRV]). As shown by [KM1], the MTW tensor associated to $c : M \times \bar{M} \rightarrow \mathbf{R}$, can be understood as the curvature of a

pseudo-metric

$$(2.1) \quad h = -\frac{1}{2}c_{i\bar{j}} \left(dx^i \otimes d\bar{x}^{\bar{j}} + d\bar{x}^{\bar{j}} \otimes dx^i \right)$$

defined on $M \times \bar{M} \setminus \text{sing}(c)$. For $(V, \bar{V}, W, \bar{W}) \in T_{(x, \bar{x})}(M \times \bar{M}) \times T_{(x, \bar{x})}(M \times \bar{M})$ (i.e. $V, W \in T_x M$, $\bar{V}, \bar{W} \in T_{\bar{x}} \bar{M}$), it is computed in local coordinates [MTW] as

$$\text{MTW}(V, \bar{V}, W, \bar{W}) = (-c_{ij\bar{k}\bar{l}} + c_{ij\bar{a}}c^{\bar{a}b}c_{\bar{k}\bar{l}b})V^iW^j\bar{V}^{\bar{k}}\bar{W}^{\bar{l}}.$$

At this point we mention the $(2, 2)$ form of the tensor, which appeared first in [T]. While the coordinate invariance implicit in [T] is easily checked directly by an elementary calculation (without referring to curvature), we mention how it comes from (2.1). Due to the structure of the metric (2.1) the sharp operator, which identifies covectors with vectors, on $M \times \bar{M}$ is actually a map from T^*M to $T\bar{M}$. Thus we have the following $(2, 2)$ tensor:

$$(2.2) \quad \text{MTW} : TM \times T^*M \times TM \times T^*M \rightarrow \mathbb{R}$$

$$\text{MTW}(V, \eta, W, \zeta) = (-c_{ij\bar{r}\bar{p}} + c_{ij\bar{s}}c^{\bar{s}m}c_{m\bar{p}\bar{r}})V^iW^jc^{\bar{p}k}c^{\bar{r}l}\eta_k\zeta_l.$$

This formulation of the tensor arises naturally in the MTW calculation and has the advantage of having quantities all in terms of vectors and covectors on M . We say that c satisfies $\text{MTW}(\delta)$ condition with respect to a metric g on M if, for $\delta > 0$

$$(2.3) \quad \text{MTW}(V, \eta, V, \eta) \geq \delta \|V\|_g^2 \|\eta\|_g^2$$

for all vector covector pairs with $\eta(V) = 0$. The metric g on the source needs nothing to do with the cost function, but by fixing the metric, (2.3) becomes invariant.

2.2. Stay away from singularity property for a cost function and c -convexity of potential functions. Fix Riemannian metrics g, \bar{g} on M, \bar{M} , respectively. We say that a cost $c : M \times \bar{M} \rightarrow \mathbf{R} \cup \{+\infty\}$ has the *stay-away-from-singularity property* if for each $0 < \lambda_1, \lambda_2$, there exists $\epsilon > 0$ depending only on λ_1, λ_2 and c such that

$$(2.4) \quad \lambda_1 \leq |\det DT| \leq \lambda_2 \implies \text{dist}(\text{graph}(T), \text{sing}(c)) \geq \epsilon$$

for any differentiable map $T : M \rightarrow \bar{M}$ given by (1.1) with C^2 locally strictly c -convex potential function $u : M \rightarrow \mathbf{R}$. Here, the C^2 potential function u is called *locally strictly c -convex* if

$$(2.5) \quad u_{ij}(x) + c_{ij}(x, T(x)) \text{ is positive definite for each } x \in M.$$

This local strict c -convexity on u is equivalent to that the map T is a local diffeomorphism. In the above, $\det DT$ is computed with respect to the metrics g and \bar{g} , and the distance function dist is with respect to the product metric $g \oplus \bar{g}$ on $M \times \bar{M}$.

To be a solution to the optimal transportation problem for ρ and $\bar{\rho}$, a solution $u : M \rightarrow \mathbf{R}$ of (1.2) has to be a *global c -convex function*. Namely, u is given as a pair (u, \bar{u}) as

$$u(x) = \sup_{\bar{x} \in \bar{M}} -c(x, \bar{x}) - \bar{u}(\bar{x}), \quad \bar{u}(\bar{x}) = \sup_{x \in M} -c(x, \bar{x}) - u(x)$$

for all $(x, \bar{x}) \in M \times \bar{M}$. If M is a closed manifold and the cost function c is locally semi-concave on the set where its value is finite, then for any C^2 locally strictly c -convex function u on M , the *global c -convexity* is implied if the corresponding map T via (1.1) is a global diffeomorphism. To see this, suppose u is not globally c -convex. Then, there exists $x_0 \in M$ and $x_1 \neq x_0$ such that

$$u(x_1) + c(x_1, T(x_0)) < u(x_0) + c(x_0, T(x_0))$$

Thus on the closed manifold M , there is an absolute minimum point $z_0 \neq x_0$, of the function $u(\cdot) + c(\cdot, T(x_0))$. Near z_0 the cost $c(\cdot, T(x_0))$ should be finite by the minimum property, thus locally semi-concave by the assumption on c . Since u is C^2 , the function $u(\cdot) + c(\cdot, \bar{y})$ is semi-concave too, hence superdifferentiable. Thus, this function cannot achieve a minimum at points of nondifferentiability, so we conclude that $u(\cdot) + c(\cdot, T(x_0))$ is differentiable at z_0 . Therefore,

$$Du(z_0) + c(z_0, \bar{y}) = 0$$

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In particular the local diffeomorphism T is not one-to-one, showing the claimed equivalence.

The stay-away-from-singularity property is shown for the round sphere $M = \bar{M} = S^n$ with $c = \text{dist}^2/2$ by Delanoë and Loeper [DL], and later also for the reflector antenna cost $c = -\log|x - \bar{x}|$ on $S^n \subset \mathbf{R}^{n+1}$ [Lo2], and for $c = \text{dist}^2/2$ on the perturbations of the round sphere and its discrete quotient [DG, LV, KM2], on the Hopf fibration quotients of the sphere such as \mathbf{CP}^n and \mathbf{HP}^n [KM2]. Recently, such result is also shown for $c = \text{dist}^2/2$ on the products of round spheres $M = \bar{M} = S_{r_1}^{n_1} \times \cdots \times S_{r_k}^{n_k}$ of arbitrary dimension and size [FKM3].

Remark 4. Strictly speaking, the stay-away results in the papers [DL, Lo2, DG, LV, KM2, FKM3] are shown with respect to globally c -convex functions, which in general differ from locally c -convex ones. However, in these cases, one can actually prove such stay-away property with respect to locally strict c -convex functions. Alternatively, it can be shown that in those cases, locally c -convex functions are globally c -convex, using the results of [TW2, V, TW3, FRV].

3. ESTIMATES

In the following, T will always denote the map given by the relation (1.1), which is equivalent in any coordinate chart $x = (x^1, \dots, x^n)$, to

$$(3.1) \quad u_i(x) + c_i(x, T(x)) = 0.$$

Further differentiation gives

$$u_{ij} + c_{ij}(x, T) + c_{i\bar{s}}(x, T)T_j^{\bar{s}} = 0.$$

Here and henceforth we use $\bar{r}, \bar{s}, \bar{p}, \bar{v}$, etc to denote differentiation in the second variable of $c(x, \bar{x})$, with the map T represented in coordinates as $T^{\bar{s}}$, $\bar{s} = 1, \dots, n$. Taking the determinant of the above equation gives the elliptic equation (1.2). A

useful observation is that the matrix $w_{ij}(x) := u_{ij}(x) + c_{ij}(x, T(x))$ (when positive definite) gives a Riemannian metric on M by the following identity:

$$(Id \times T)^* h \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = w_{ij}$$

where $(Id \times T)^* h$ is the pull-back of h by $Id \times T : M \ni x \mapsto (x, T(x)) \in M \times M$.

For future reference we note the following results of differentiating

$$(3.2) \quad T_j^{\bar{s}} = -c^{i\bar{s}} w_{ij},$$

$$(3.3) \quad T_{jk}^{\bar{s}} = -c^{i\bar{s}} \frac{\partial}{\partial x^k} w_{ij} + c^{i\bar{p}} c^{l\bar{s}} (c_{l\bar{p}k} + c_{l\bar{p}\bar{r}} T_k^{\bar{r}}) w_{ij}.$$

Here, to be clear, we write $\frac{\partial}{\partial x^k} w_{ij}$ to express coordinate derivatives of this tensor. Subscripts for other functions mean the corresponding coordinate derivatives. Superscripts such as $c^{i\bar{s}}$ denotes the (i, \bar{s}) -entry of the inverse matrix of $\left(\frac{\partial^2}{\partial x^a \partial \bar{x}^b} c \right)$, etc. Defining

$$(3.4) \quad \theta(x, u) = \ln \det w_{ij} - \ln \rho(x) + \ln \bar{\rho}(T(x)) - \ln |\det c_{i\bar{s}}(x, T(x))|,$$

the flow (1.3) is rewritten as

$$(3.5) \quad u_t = \theta.$$

Here and henceforth, the subscript t denotes the time derivative.

3.1. Stay away from singularity. We make use of the linearized operator L of θ at a function u . This operator L is covariant, and as seen from [TW] the operator L has the following local expression.

$$(3.6) \quad Lv = w^{ij} v_{ij} - (w^{ij} c_{ij\bar{s}} c^{\bar{s}k} - (\bar{\rho})^{-1} \bar{\rho}_{\bar{r}} c^{k\bar{r}} - c^{i\bar{s}} c_{\bar{s}i\bar{p}} c^{\bar{p}k}) v_k.$$

where all coefficients are computed at $(x, T(x))$. It is important to notice that there is no zeroth-order term in L so that we can apply parabolic maximum principle.

Lemma 3.1. *If u is a solution of (3.5), then θ satisfies*

$$(3.7) \quad \theta_t = L\theta.$$

Proof. This follows from the definition of linearized operator $\frac{\partial}{\partial t} (\theta(x, u)) = L \frac{\partial}{\partial t} u$ and using the equation (3.5). \square

Proposition 3.2 (Stay-away-from-singularity property along the flow). *Let M, \bar{M} be two n -dimensional compact Riemannian manifolds without boundary, equipped with cost function $c : M \times \bar{M} \rightarrow \mathbf{R}$ satisfying (1.4) and (1.5). Let $\rho, \bar{\rho}$ be smooth, positive, probability densities on M, \bar{M} , respectively. Suppose the cost c satisfies the stay-away-from-singularity property (see Section 2.2). There exists a constant*

$$\varepsilon = \varepsilon(\theta|_{t=0}, \|\ln \rho\|_\infty, \|\ln \bar{\rho}\|_\infty) > 0,$$

such that in the time interval of existence of the smooth solution u to the parabolic equation (1.3), $\text{dist}(\text{graph}(T), \text{sing}(c)) \geq \varepsilon$ for T given by (1.1). Here, ε is independent of the time t .

Proof. It follows from Lemma 3.1 and the parabolic maximum principle applied to (3.7) that θ is bounded along the flow. Now as

$$\theta(x, t) = \ln \det DT - \ln \rho(x) - \ln \bar{\rho}(T(x)),$$

a bound on $\|\ln \det DT\|_\infty$ follows from the bound on θ . This together with the stay-away-property (2.4) completes the proof. \square

Corollary 3.3 (Uniform bounds on derivatives of c along the flow). *Suppose the same assumption as in Proposition 3.2 holds. Then, in the time interval of existence of the smooth solution u to the flow of (1.3), each derivative of c in any order such as $c_{ij}, c_{i\bar{s}}, c_{ijk}, c_{ij\bar{k}}$, etc. at $(x, T(x))$ computed in any fixed coordinates, is uniformly bounded, especially independent of t . Moreover, $c_{i\bar{s}}$ is uniformly away from 0.*

3.2. Second derivative estimates. On the both source and target manifolds, we work in normal coordinates throughout this section. Noting that the transition functions between charts change in a predictable way, an $\varepsilon/4$ covering argument can be used to show there is a uniform bound on (in particular third and fourth) derivatives of the cost function which doesn't depend on the charts, regardless of the point at the origin. Thus throughout this subsection, we assume the result of Corollary 3.3, which is an essential ingredient in the following estimates. Following the calculations given in [MTW] we show

Lemma 3.4. *Assume the uniform bounds in Corollary 3.3. Let u be a solution of (3.5), and let v be a coordinate direction in a local chart. Then*

$$(3.8) \quad \begin{aligned} \frac{\partial}{\partial t} w_{vv} - (Lw)_{vv} &\leq -w^{ij} (-c_{ij\bar{p}\bar{r}} + c_{ij\bar{s}} c^{\bar{s}m} c_{m\bar{r}\bar{p}}) c^{\bar{p}k} c^{\bar{r}l} w_{lv} w_{kv} \\ &\quad + C(\rho, \bar{\rho}, D^4 c, g) (1 + \sum w_{ii}^2 + \sum w^{ii} \sum w_{jj}). \end{aligned}$$

Here L is operating on the $(2, 0)$ tensor using metric covariant differentiation.

Proof. To simplify the calculation choose normal coordinates around x such that $v = \frac{\partial}{\partial x_1}$. Now from (3.6),

$$(Lw)_{11} = w^{ij} (w_{11,ij} - w_{ij,11}) + w^{ij} w_{ij,11} - (w^{ij} c_{ij\bar{s}} c^{\bar{s}k} - (\bar{\rho})^{-1} \bar{\rho}_{\bar{r}} c^{k\bar{r}} - c^{i\bar{s}} c_{\bar{s}i\bar{p}} c^{\bar{p}k}) \partial_k w_{11},$$

with

$$\begin{aligned} w_{ij,11} &= u_{ij11} + c_{ij11} + 2c_{ij\bar{s}1} T_1^{\bar{s}} + c_{ij\bar{s}} T_{11}^{\bar{s}} + c_{ij\bar{p}\bar{r}} T_1^{\bar{p}} T_1^{\bar{r}} + w * \partial^2 g \\ w_{11,ij} &= u_{11ij} + c_{11ij} + c_{11\bar{s}i} T_j^{\bar{s}} + c_{11\bar{s}j} T_i^{\bar{s}} + c_{11\bar{s}} T_{ij}^{\bar{s}} + c_{11\bar{p}\bar{r}} T_i^{\bar{p}} T_j^{\bar{r}} + w * \partial^2 g \end{aligned}$$

where $w * \partial^2 g$ denotes a linear combination of second derivatives of the metric with components of w . Thus

$$\begin{aligned} (Lw)_{11} &\geq -K + w^{ij} (c_{11\bar{s}} T_{ij}^{\bar{s}} + c_{11\bar{p}\bar{r}} T_i^{\bar{p}} T_j^{\bar{r}} - c_{ij\bar{s}} T_{11}^{\bar{s}} - c_{ij\bar{p}\bar{r}} T_1^{\bar{p}} T_1^{\bar{r}}) \\ &\quad + w^{ij} \partial_{11} w_{ij} - (w^{ij} c_{ij\bar{s}} c^{\bar{s}k} - (\ln \bar{\rho})_{\bar{r}} c^{k\bar{r}} - c^{i\bar{s}} c_{\bar{s}i\bar{p}} c^{\bar{p}k}) \partial_k w_{11}. \end{aligned}$$

Here as in [MTW] we let

$$K = C \sum w^{ii} \sum w_{jj} + C \sum w^{ii} + C \sum w_{ii}^2.$$

Now use (3.3) to see

$$(3.9) \quad T_{11}^{\bar{s}} = -c^{k\bar{s}}w_{k1,1} + c^{k\bar{p}}c^{l\bar{s}}(c_{l\bar{p}1} + c_{l\bar{p}\bar{r}}T_1^{\bar{r}})w_{k1}$$

and

$$(3.10) \quad w^{ij}T_{ij}^{\bar{s}} = -c^{k\bar{s}}w^{ij}(\partial_j w_{ki} - \partial_k w_{ij}) - c^{k\bar{s}}w^{ij}\partial_k w_{ij} + c^{j\bar{p}}c^{l\bar{s}}(c_{l\bar{p}j} + c_{l\bar{p}\bar{r}}T_j^{\bar{r}}).$$

Note that

$$\partial_j w_{ki} - \partial_k w_{ij} = c_{k\bar{i}\bar{s}}T_j^{\bar{s}} - c_{i\bar{j}\bar{s}}T_k^{\bar{s}}$$

and use this to cancel one $\partial_k w_{11}$ term to get

$$(Lw)_{11} \geq -K - c_{11\bar{s}}c^{k\bar{s}}w^{ij}\partial_k w_{ij} + w^{ij}(-c_{ij\bar{s}}c_{l\bar{p}\bar{r}}T_1^{\bar{r}}w_{k1} - c_{ij\bar{p}\bar{r}}T_1^{\bar{p}}T_1^{\bar{r}}) \\ + w^{ij}\partial_{11}w_{ij} + ((\ln \bar{\rho})_{\bar{r}}c^{k\bar{r}} + c^{i\bar{s}}c_{\bar{s}i\bar{p}}c^{\bar{p}k})\partial_k w_{11}.$$

Now using

$$\theta_{11} = w^{ij}\partial_{11}w_{ij} + \partial_1 w^{ij}\partial_1 w_{ij} + (\ln \bar{\rho})_{\bar{s}}T_{11}^{\bar{s}} + c^{i\bar{s}}c_{i\bar{s}\bar{p}}T_{11}^{\bar{p}} + c^{i\bar{s}}c_{i\bar{s}\bar{p}\bar{r}}T_1^{\bar{p}}T_1^{\bar{r}} - K \\ \geq w^{ij}\partial_{11}w_{ij} + ((\ln \bar{\rho})_{\bar{s}} + c^{i\bar{s}}c_{i\bar{s}\bar{p}})c^{k\bar{s}}w_{k1,1} - K$$

we have

$$(Lw)_{11} \geq -K - c_{11\bar{s}}c^{k\bar{s}}\theta_k + \theta_{11} \\ - w^{ij}(-c_{ij\bar{s}}c^{l\bar{s}}c_{l\bar{p}\bar{r}} + c_{ij\bar{p}\bar{r}})T_1^{\bar{r}}T_1^{\bar{p}}$$

Now, finally we compute

$$\frac{\partial}{\partial t}w_{11} = \theta_{11} + c_{11\bar{s}}T_t^{\bar{s}}.$$

Differentiating (3.1) with respect to t we get

$$u_{kt} + c_{k\bar{s}}T_t^{\bar{s}} = 0,$$

which, subtracting, we arrive at

$$(Lw)_{11} - w_{11t} \geq -w^{ij}(-c_{ij\bar{s}}c^{l\bar{s}}c_{l\bar{p}\bar{r}} + c_{ij\bar{p}\bar{r}})T_1^{\bar{r}}T_1^{\bar{p}} - K$$

which is the conclusion of the Lemma using (3.2). \square

Theorem 3.5. *For a closed Riemannian manifold M and an interval $[0, l]$, suppose that $u : M \times [0, l] \rightarrow \mathbf{R}$ is a smooth solution to the parabolic equation (3.5). Assume that c satisfies (i) the MTW(δ) condition, and (ii) the stay-away-from-singularity property (so that the uniform bounds of Corollary 3.3 hold on $M \times [0, l]$). Let $v \in TM$, $|v| = 1$, be such that*

$$w_{vv} = \max_{z \in TM, |z|=1} w_{zz}.$$

There is a constant $C = C(\delta, c, n, g)$, especially independent of t , such that if $w_{vv}(x, t) \geq C$ then

$$w_{vvt}(x, t) \leq 0.$$

In particular, $w_{zz}(x, t) \leq C$ for any $(x, t) \in M \times [0, l]$ and $|z| = 1$.

Proof. We use Hamilton's [Ha] parabolic maximum principle argument for tensors. Analyze the first term in the right hand side of (3.8)

$$(3.11) \quad -w^{ij} \text{MTW}_{ij}^{kl} w_{lv} w_{kv}$$

where

$$\text{MTW}_{ij}^{kl} = (-c_{ij\bar{p}\bar{r}} + c_{ij\bar{s}} c^{\bar{s}m} c_{m\bar{r}\bar{p}}) c^{\bar{p}k} c^{\bar{r}l}.$$

Diagonalize w with $v = \frac{\partial}{\partial x_1}$, and (3.11) becomes

$$-w^{11} \text{MTW}_{11}^{11} w_{11}^2 - \sum_{i>1} w^{ii} \text{MTW}_{ii}^{11} w_{11}^2.$$

By the MTW (δ) condition, we have

$$\sum_{i>1} w^{ii} \text{MTW}_{ii}^{11} w_{11}^2 \geq \sum_{i>1} w^{ii} \delta \|\partial x_i\|_g^2 \|w_{11} dx_1\|_g^2,$$

thus (3.11) is bounded by

$$(3.12) \quad C \sum w_{ii} - \delta \sum_{i>1} w^{ii} w_{11}^2 \leq \sum w_{ii} - \delta c_n \sum_i w^{ii} w_{11}^2$$

as our chart for the source M is normal at x_0 . Since $\det w_{ij}$ is bounded it follows from the arithmetic-geometric mean that

$$c \left(\sum w_{ii} \right)^{1/(n-1)} \leq \sum w^{ii} \leq C \left(\sum w_{ii} \right)^{n-1}.$$

So finally, from Lemma 3.4

$$\begin{aligned} \frac{\partial}{\partial t} w_{vv} - (Lw)_{vv} &\leq C \sum w_{ii} - \delta c_n \sum_i w^{ii} w_{11}^2 \\ &\quad + C(1 + \sum w_{ii}^2 + \sum w^{ii} \sum w_{jj}) \\ &\leq -\frac{\delta}{2} c_n \sum_i w^{ii} w_{11}^2 + C \sum w_{ii} + C \sum w_{ii}^2 \\ &\quad + C \sum w^{ii} \left(\sum w_{jj} - \frac{\delta}{2} c_n w_{11}^2 \right) \\ &\leq -\frac{\delta}{2} c_n \left(\sum w_{ii} \right)^{1/(n-1)} c_n \left(\sum w_{ii} \right)^2 + C \left(\sum w_{ii} \right)^2 \\ &\quad + C \sum w^{ii} \left\{ \sum w_{ii} - \frac{\delta}{2} c_n \left(\sum w_{ii} \right)^2 \right\}. \end{aligned}$$

We see that when $\sum w_{ii}$ is sufficiently large, the right-hand side must be negative. This completes the proof. \square

Corollary 3.6. *In the situation of Theorem 3.5, the spatial second derivatives of the solution u to the parabolic equation (1.3) remain bounded (uniformly in time) and u stays locally strictly c -convex.*

Proof. An upper bound on the eigenvalues of w_{ij} is given in Theorem 3.5. From the identity $w_{ij} = u_{ij} + c_{ij}$, the bound on u_{ij} follows. Because θ remains bounded by parabolic maximum principle for $\theta_t = L\theta$, we have a lower bound on determinant

$\det w_{ij}$. An upper bound on eigenvalues, plus lower bound on determinant implies lower bound on the eigenvalues. It follows that

$$u_{ij} + c_{ij} \geq \varepsilon g_{ij}.$$

for some $\varepsilon > 0$, thus local strict c -convexity of u follows. \square

4. PROOF OF THEOREM 1.1: EXISTENCE OF SOLUTION AND UNIFORM BOUNDS

In this section we show that the solution to parabolic equation (1.3) exists for all $t \in \mathbf{R}_+$ under the assumptions of Theorem 1.1. We also show the solution has uniform C^k derivatives in x , where each C^k norm is uniform in t . Through this section we use Corollary 3.3 in an essential way. In the following, all the $C^{1,1}, C^{2,\alpha}, C^{4,\alpha}$ estimates and so on, are estimates on the derivatives in x , and are all uniform in the time variable t . But, by C^α we will mean C^α both in x and t .

Short-time existence: Since M is a closed manifold, a standard theory implies the existence of a short-time solution to (1.3) for any locally strictly c -convex smooth initial data, and that from Corollary 3.6 the solution u is locally strictly c -convex on the time interval of existence. (See [Kt] for a proof of short time existence regarding the same equation with a more involved boundary condition.)

Long-time existence: Apply Theorem 3.5 (or Corollary 3.6) to get $C^{1,1}$ estimates for u . This in particular makes the equation (1.3) as well as the linearized equation $v_t = Lv$ (see (3.6)) uniformly parabolic with bounded coefficients. Now applying Krylov-Safonov theory (c.f. [Lb, Lemma 14.6]) to (1.3) one has $C^{2,\alpha}$ estimates. From the short-time existence above and Arzela-Ascoli, this shows that the solution cannot cease to exist at a finite time, thus exists for all $t \in [0, \infty)$.

Uniform C^k bounds: To see uniform C^k bounds, first differentiate (1.3) with respect to any coordinate direction $\frac{\partial}{\partial x^k}$, to see u_k satisfies

$$(4.1) \quad u_{kt} = Lu_k.$$

From $C^{2,\alpha}$ estimates of u , this linear equation for u_k is uniformly parabolic with C^α controlled coefficients (here C^α control in t follows from the parabolic equations like (4.1) with space $C^{2,\alpha}$ estimates), and in particular, we can apply parabolic Schauder estimates to conclude that u_k has $C^{2,\alpha}$ estimates, thus obtaining $C^{3,\alpha}$ estimates for u . Similarly, differentiating (4.1) we obtain a parabolic equation for u_{kl} also with coefficients and inhomogeneous terms all of which are C^α , and thus follows $C^{4,\alpha}$ estimates of u , so forth. Thus, we have uniform C^k , $k = 0, 1, \dots$, bounds for u as claimed.

To obtain the exponential convergence, we need a Harnack inequality, which is shown in the next sections.

5. A LI-YAU TYPE HARNACK INEQUALITY

In this section, as a preliminary step to the proof of the exponential convergence to a solution of the elliptic equation (1.2), we derive a Harnack inequality for the quantity θ . We first find a simple expression of the linearized operator L of θ . This expression allows us to derive a Harnack type estimate (see Theorem 5.2), whose corollary (Corollary 5.3) is used to show the exponential convergence in Section 7.

Let us find a convenient expression for the linearized operator L , using [Wr, Prop 2.1]. As a linearized operator of coordinate invariant fully nonlinear equation (1.3), L is expected to be related to a Laplacian operator of certain Riemannian metric. More precisely, for a manifold M , suppose that a map T is c -exp du for some u . (This means by definition T is given by (1.1).) Define a function ψ by

$$(5.1) \quad \psi = \left(\frac{\bar{\rho}^2(T(x)) \det DT}{|\det c_{i\bar{s}}(x, T(x))|} \right)^{1/(n-2)}.$$

We observe the following

Proposition 5.1. *Let $n \geq 3$. The linearized operator L of θ (see (3.4)) is expressed as*

$$Lv = \psi \Delta_{\psi w} v,$$

where $\Delta_{\psi w}$ is the Laplace-Beltrami operator with respect to the metric ψw_{ij} given on M .

Proof. First of all, let

$$(5.2) \quad g_{ij}(x) = w_{ij}(x) \left(\frac{\rho(x) \bar{\rho}(T(x))}{|\det c_{i\bar{s}}(x, T(x))|} \right)^{1/(n-2)}.$$

Also, recall

$$\theta = \ln \det DT - \ln \rho(x) + \ln \bar{\rho}(T(x)).$$

Then [Wr, Prop 2.1] states that

$$Lv = \left(\frac{\rho(x) \bar{\rho}(T(x))}{\det c_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)} \left(\Delta_g v + \frac{1}{2} \langle \nabla v, \nabla \theta \rangle_g \right).$$

A general formula for conformal metrics shows that if $\tilde{g} = e^f g$, then

$$e^f \tilde{\Delta} \phi = \Delta \phi + \frac{n-2}{2} \langle \nabla \phi, \nabla f \rangle.$$

It follows immediately that

$$Lv = \left(\frac{\rho(x) \bar{\rho}(T(x))}{\det c_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)} e^{\theta/(n-2)} \Delta_{e^{\theta/(n-2)} g} v.$$

Using the expression for θ we observe that

$$\begin{aligned} & \left(\frac{\rho(x)\bar{\rho}(T(x))}{\det c_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)} e^{\theta/(n-2)} \\ &= \left(\frac{\rho(x)\bar{\rho}(T(x))}{\det c_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)} \left(\frac{(\det DT)\bar{\rho}(T(x))}{\rho(x)} \right)^{1/(n-2)} \\ &= \left(\frac{\bar{\rho}^2(T(x)) \det DT}{\det c_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)}. \end{aligned}$$

The result follows. \square

Noting that by Lemma 3.1, and Proposition 5.1

$$\theta_t = L\theta = \psi \Delta_{\psi w} \theta$$

we derive a Harnack estimate for the operator L , when $n \geq 3$. Here, the expression $\psi \Delta_{\psi w}$ enables us to easily modify the argument in [LY] to obtain

Theorem 5.2 (Harnack inequality). *Let M be a compact manifold of dimension n and let $g(t)$, $0 \leq t < \infty$ be a family of Riemannian metrics on M such that*

$$(5.3) \quad \begin{aligned} \frac{1}{C_0} g(0) &\leq g(t) \leq C_0 g(0) \\ \left| \frac{\partial}{\partial t} g \right| &\leq C_0 \\ R_{ij}(t) &\geq -K g_{ij}(0) \end{aligned}$$

with universal constants $C_0, K > 0$. Let $\lambda(x, t)$ be a positive function with derivatives uniformly controlled (independent of (t, x)) and bounded above and away from zero. Let $U(x, t)$ be a nonnegative solution to

$$(5.4) \quad U_t = \lambda(x, t) \Delta_{g(t)} U.$$

Then there exists a constant $C > 0$ depending only on $C_0, K, g_{ij}(0)$ and the bounds on the derivatives of λ so that for $0 < t_1 < t_2 < \infty$

$$\sup_{x \in M} U(x, t_1) \leq \inf_{x \in M} U(x, t_2) C \frac{t_2}{t_1} \exp \left\{ \frac{C_0^2 \text{diam}^2(M_{t_1})}{(t_2 - t_1)} + C(t_2 - t_1) \right\}.$$

Note that by the a priori estimates in the previous sections, the metric $g_{ij} = \psi w_{ij}$ and scalar $\lambda = \psi$ all satisfy the assumptions in Theorem 5.2, thus we obtain the corresponding Harnack inequality for the operator L . In particular, we have

Corollary 5.3. *For $n \geq 3$, let $U : M \times [0, \infty] \rightarrow \mathbf{R}$ be a solution to the parabolic equation $U_t = LU$, where L is the linearized operator in (3.6). There exists a constant independent of $(x, t) \in M \times [0, \infty]$ such that*

$$\sup_{x \in M} U(x, t + 1/2) \leq C \inf_{x \in M} U(x, t).$$

6. PROOF OF THEOREM 5.2 (HARNACK INEQUALITY)

This whole section is devoted to the proof of the Harnack inequality claimed in Theorem 5.2. Since the equation (5.4) is not a pure heat equation, but conformally related to one for a time dependant metric, we are forced to reprove the Harnack estimate for this operator. The following argument is a slight modification of that found in [LY]. Let

$$f = \log U,$$

then

$$f_t = \lambda(\Delta f + |\nabla f|^2).$$

Now let

$$F = t(\lambda|\nabla f|^2 - \alpha f_t).$$

We directly compute

$$\begin{aligned} \Delta F &\geq t \left(\begin{aligned} &\lambda \Delta |\nabla f|^2 + 2 \langle \nabla \lambda, \nabla |\nabla f|^2 \rangle + \Delta \lambda |\nabla f|^2 - \alpha (\Delta f)_t \\ &- \alpha \left| \frac{\partial g}{\partial t} \right| (|\nabla^2 f| + |\nabla f|) \end{aligned} \right) \\ &\geq t \left(\begin{aligned} &2\lambda \|\nabla^2 f\|^2 + 2\lambda \langle \nabla f, \nabla \Delta f \rangle - 2\lambda K |\nabla f|^2 - \varepsilon \|\nabla^2 f\|^2 - \frac{C_1}{\varepsilon} |\nabla f|^2 \\ &- C_2 |\nabla f|^2 - \alpha (\Delta f)_t - \varepsilon \|\nabla^2 f\|^2 - \frac{C_3}{\varepsilon} \end{aligned} \right) \\ &\geq t \left(\gamma \|\nabla^2 f\|^2 + 2\lambda \langle \nabla f, \nabla \Delta f \rangle - \alpha (\Delta f)_t - C_4 |\nabla f|^2 - C_3 \right). \end{aligned}$$

In the second line we applied the Bochner formula, with K a uniform lower bound on the time-dependent Ricci curvature, C_1 is an upper bound on $\|\nabla \lambda\|$, ε is somewhat smaller than a lower bound on λ , C_2 bounds $\Delta \lambda$ and the constant C_3 bounds the time derivative of the metric. The constant γ is a positive lower bound for λ .

Now

$$(6.1) \quad \Delta f = -|\nabla f|^2 + \frac{1}{\lambda} f_t = -\frac{1}{\lambda} \left(\frac{1}{t} F - (1 - \alpha) f_t \right).$$

We thus further estimate

$$\begin{aligned} \Delta F &\geq t \left(\begin{aligned} &\gamma \|\nabla^2 f\|^2 - 2\lambda \langle \nabla f, \nabla \frac{1}{\lambda} \left(\frac{1}{t} F - (1 - \alpha) f_t \right) \rangle + \\ &\alpha \left[\frac{1}{\lambda} \left(\frac{1}{t} F - (1 - \alpha) f_t \right) \right]_t - C_4 |\nabla f|^2 - C_3 \end{aligned} \right) \\ &\geq t \gamma \|\nabla^2 f\|^2 - 2 \langle \nabla f, \nabla F \rangle + 2t(1 - \alpha) \langle \nabla f, \nabla f_t \rangle - C_5 t |\nabla f| |(\lambda \Delta f)| \\ &\quad + \frac{1}{\lambda} \left(\alpha F_t - \alpha \frac{F}{t} - t \alpha (1 - \alpha) f_{tt} \right) - C_6 t |(\lambda \Delta f)| - C_4 t |\nabla f|^2 - C_3 t \end{aligned}$$

$$\begin{aligned}
&\geq t\gamma_2 \|\nabla^2 f\|^2 - 2\langle \nabla f, \nabla F \rangle + \frac{1}{\lambda} \{2t\lambda(1-\alpha)\langle \nabla f, \nabla f_t \rangle - t\alpha(1-\alpha)f_{tt}\} \\
&+ \frac{1}{\lambda} \left(\alpha F_t - \alpha \frac{F}{t} \right) - C_7 t |\nabla f|^2 - C_3 t \\
&= t\gamma_2 \|\nabla^2 f\|^2 - 2\langle \nabla f, \nabla F \rangle + \frac{1}{\lambda} \left\{ (1-\alpha)F_t - (1-\alpha)\frac{F}{t} - \lambda t C_3 |\nabla f|^2 \right\} \\
&+ \frac{1}{\lambda} \left(\alpha F_t - \alpha \frac{F}{t} \right) - C_7 t |\nabla f|^2 - C_3 t \\
&\geq t\gamma_2 \|\nabla^2 f\|^2 - 2\langle \nabla f, \nabla F \rangle + \frac{1}{\lambda} F_t - \frac{1}{\lambda} \frac{F}{t} - tC_9 |\nabla f|^2 - C_3 t \\
&\geq t \frac{\gamma_2}{\lambda n} (-\lambda |\nabla f|^2 + f_t)^2 - 2\langle \nabla f, \nabla F \rangle + \frac{1}{\lambda} F_t - \frac{1}{\lambda} \frac{F}{t} - tC_9 |\nabla f|^2 - C_3 t.
\end{aligned}$$

Here C_5 bounds $|\nabla \frac{1}{\lambda}|$, and C_6 bounds $|\frac{1}{\lambda}|_t$, with γ_2 an even smaller constant, related to C_7 . In the last line we applied $\|\nabla^2 f\|^2 \geq \frac{1}{n}(\Delta f)^2$ and the relation (6.1).

Now at a maximum for F which happens at some positive time on $M \times [0, T]$ we can conclude

$$\begin{aligned}
0 &\geq t \frac{\gamma_2}{\lambda n} (-\lambda |\nabla f|^2 + f_t)^2 - 2\langle \nabla f, \nabla F \rangle + \frac{1}{\lambda} F_t - \frac{1}{\lambda} \frac{F}{t} - tC_9 |\nabla f|^2 - C_3 t \\
(6.2) \quad &\geq t\gamma_3 (-\lambda |\nabla f|^2 + f_t)^2 - \frac{1}{\lambda} \frac{F}{t} - tC_9 |\nabla f|^2 - C_3 t.
\end{aligned}$$

Now let

$$\begin{aligned}
y &= \lambda |\nabla f|^2, \\
z &= f_t.
\end{aligned}$$

Continuing to follow [LY, Eq 1.9] we expand

$$(6.3) \quad (y-z)^2 = \left(\frac{1}{\alpha} - \frac{\varepsilon}{2} \right) (y-\alpha z)^2 + (1-\varepsilon-\delta - \frac{1}{\alpha} + \frac{\varepsilon}{2}) y^2 + (1-\alpha + \frac{\varepsilon}{2} \alpha^2) z^2 + \varepsilon y(y-\alpha z) + \delta y^2$$

Choose

$$\begin{aligned}
\varepsilon &= 2 - 2\frac{1}{\alpha} - 2\frac{1}{(1-\alpha)^2} > 0 \\
\delta &= \frac{1}{(1-\alpha)^2}
\end{aligned}$$

and α so that

$$\begin{aligned}
\frac{1}{\alpha} &> \varepsilon > 0 \\
\delta &> \frac{\varepsilon}{2} > 0.
\end{aligned}$$

Specifically, one may choose $\alpha = \frac{5}{2}$ to satisfy these conditions. Note that the second and third terms in (6.3) vanish, and multiplying (6.2) by t , absorbing bounds on λ

and combining with (6.3)

$$\begin{aligned}
0 &\geq t^2 \gamma_4 \left\{ \left(\frac{1}{\alpha} - \frac{\varepsilon}{2} \right) (y - \alpha z)^2 + \varepsilon y(y - \alpha z) + \delta y^2 \right\} - F - t^2 C_{10} y - C_{11} t^2 \\
&\geq \gamma_4 \left\{ \left(\frac{1}{\alpha} - \frac{\varepsilon}{2} \right) F^2 - \frac{\varepsilon}{2} F^2 - \frac{\varepsilon}{2} t^2 y^2 + \delta t^2 y^2 \right\} - F - t^2 C_{10} y - C_{11} t^2 \\
&\geq \gamma_4 \left(\frac{1}{\alpha} - \varepsilon \right) F^2 + \gamma_4 \left(\delta - \frac{\varepsilon}{2} \right) t^2 y^2 - F - t^2 \left[\frac{(C_2 + K)}{\gamma_4 \left(\delta - \frac{\varepsilon}{2} \right)} + \gamma_4 \left(\delta - \frac{\varepsilon}{2} \right) y^2 \right] - C_{11} t^2.
\end{aligned}$$

From which we conclude that

$$F \leq \frac{1}{2\gamma_4 \left(\frac{1}{\alpha} - \varepsilon \right)} + \frac{\sqrt{1 + \left(4 \frac{C_{10}}{\gamma_4 \left(\delta - \frac{\varepsilon}{2} \right)} + C_{11} \right) \gamma_4 \left(\frac{1}{\alpha} - \varepsilon \right) t^2}}{\gamma_4 \left(\frac{1}{\alpha} - \varepsilon \right)}$$

i.e

$$(6.4) \quad \lambda |\nabla f|^2 - \alpha f_t \leq \frac{C}{t} + C.$$

Now we have arrived at this conclusion assuming that the maximum happens away from $t = 0$. Letting $f = \log(U + \iota)$ this assumption is available. We then take $\iota \rightarrow 0$.

Now consider now a path $\gamma : [0, 1] \rightarrow M \times [t_1, t_2]$ such that $\gamma(0) = (y, t_2)$ and $\gamma(1) = (x, t_1)$, which for convenience, projects to a geodesic in M_{t_1} and has constant speed in t . Using the assumption (5.3) and (6.4)

$$\begin{aligned}
f(x, t_1) - f(y, t_2) &\leq \int_0^1 \{ |\nabla f| C_0 d(x, y) - (t_2 - t_1)(f)_t \} ds \\
&\leq \int_0^1 \left\{ |\nabla f| C_0 d(x, y) + (t_2 - t_1) \left(C + \frac{C}{t} - \frac{\lambda |\nabla f|^2}{\alpha} \right) \right\} ds \\
&\leq \int_0^1 \left\{ \frac{\alpha}{4(t_2 - t_1)\lambda} C_0^2 d^2(x, y) + (t_2 - t_1) \left(C + \frac{C}{t} \right) \right\} ds \\
&\leq \frac{\alpha}{4(t_2 - t_1) \min \lambda} C_0^2 d^2(x, y) + C(t_2 - t_1) + C \log \frac{t_2}{t_1}
\end{aligned}$$

where in the last line we integrate using that $t = (1 - s)t_2 + st_1$. This completes the proof of Harnack inequality (Theorem 5.2).

7. PROOF OF THEOREM 1.1: EXPONENTIAL CONVERGENCE TO THE SOLUTION TO OPTIMAL TRANSPORT PROBLEM

This section completes the proof of Theorem 1.1. We assume the conditions in Theorem 1.1. In the first subsection we show the exponential convergence to a stationary solution, and then in the last subsection we show that the stationary solution is indeed the solution to the optimal transport problem.

7.1. Exponential convergence.

7.1.1. *Case $n \geq 3$.* With this parabolic Harnack inequality at hand, the claimed exponential convergence to the optimal transportation map follows from a rather standard argument. To see this, consider for $k \in \mathbf{N}$,

$$U_k = \sup_{x \in M} \theta(x, k-1) - \theta(x, (k-1) + t),$$

$$L_k = \theta(x, k-1+t) - \inf_{x \in M} \theta(x, k-1)$$

that are also solutions to the equation $U_t = LU$. By the strong maximum principle, both U and L are positive functions for positive t , for all k . Also, let $H(t) = \sup_{x \in M} \theta(x, t) - \inf_{x \in M} \theta(x, t)$. The Harnack inequality in Theorem 5.2 yields

$$\sup \theta(x, k-1) - \inf \theta(x, k - \frac{1}{2}) \leq C (\sup \theta(x, k-1) - \sup \theta(x, k)),$$

$$\sup \theta(x, k - \frac{1}{2}) - \inf \theta(x, k-1) \leq C (\inf \theta(x, k) - \inf \theta(x, k-1))$$

for some fixed constant $C > 1$. It follows by adding the equations together that

$$H(k-1) + H(k - \frac{1}{2}) \leq C (H(k-1) - H(k))$$

which implies

$$H(k) \leq \epsilon H(k-1)$$

where $\epsilon = \frac{C-1}{C} < 1$. By induction we observe

$$H(k) \leq \epsilon^k H(0).$$

It follows that $H(t) \leq C e^{-\beta t}$ where $\epsilon = e^{-\beta}$. Therefore, θ converges to the limit $\theta_\infty \equiv \text{const.}$ exponentially fast as $t \rightarrow \infty$. Now the quantity θ can be larger than 0 somewhere, only if it is smaller than 0 somewhere, as can be seen by integrating the change of measures and using $\int_M \rho(x) dx = \int_{\bar{M}} \bar{\rho}(\bar{x}) d\bar{x}$. It follows that $\theta_\infty \equiv 0$ and thus from $u_t = \theta$, we see $u_t \rightarrow 0$ exponentially fast as $t \rightarrow \infty$. This implies u converges exponentially fast to a stationary solution u_∞ , which is smooth from the uniform C^m , $m \in \mathbf{N}$, estimates. Because $\theta_\infty \equiv 0$, u_∞ solves the elliptic equation (1.2). Considering the discussion in Section 7.2, this solution is a solution to the optimal transportation problem. This finishes the proof of the claimed exponential convergence and of Theorem 1.1 for $n \geq 3$.

7.1.2. *Case $n = 2$.* On $M^2 \times \bar{M}^2$ a solution u will satisfy all of the estimates, hence exists and enjoys subsequential convergence at infinity. The only missing piece is the Harnack inequality. However, we can fake a third dimension and get a solution on $M^2 \times S^1 \rightarrow \bar{M}^2 \times S^1$ by letting $\tilde{u}(x, z) = u(x)$ and taking products of the measures with uniform measures on S^1 . Hence $\tilde{u}(x, z)$ will also be a solution, and will converge in the same way to the three dimensional product solution.

7.2. The limit stationary solution is the solution to the optimal transport problem: strict global c -convexity: To conclude that the limit stationary solution, say u_∞ , is a solution to the optimal transport problem (and not a spurious solution to the elliptic equation), it remains to show that u_∞ is globally strict c -convex, which from the discussion in the middle of Section 2.2 follows if the corresponding map, say T_∞ , is a global diffeomorphism. (This map T_∞ is already a

local diffeomorphism by local strict c -convexity of u_∞ .) To see this, we use that u_∞ satisfies the Monge-Ampère type equation (1.2). If T_∞ is not one-to-one, as a local diffeomorphism between closed manifolds, it is a covering map, having the topological degree greater than 1. Thus, from (1.2) the push-forward $T_{\infty\#}\rho$ is a multiple of the target measure $\bar{\rho}$. But, this contradicts the fact that $\int_M \rho = \int_{\bar{M}} \bar{\rho}$ since $\int_{\bar{M}} T_{\infty\#}\rho = \int_M \rho$. This finishes the proof of the fact that the limiting stationary solution of (1.3) is the solution to the optimal transport problem, and thus together with all the previous sections (especially Section 4) it completes the proof of Theorem 1.1.

As a final remark, we state a corollary to Theorem 1.1, in particular this last paragraph:

Corollary 7.1. *Assume that the same conditions as in Theorem 1.1 hold. Then, any locally strictly c -convex C^2 function u_0 is in fact globally c -convex.*

Proof. Let T_0 denote the corresponding map to u_0 by the formula (1.1). Local strict c -convexity implies that T_0 is local diffeomorphism and for global c -convexity of u_0 , it is enough to show that T_0 has topological degree 1. From Theorem 1.1, the map T depends continuously on the time variable t . In particular, the degree stays constant. From the result of Theorem 1.1 the map T_∞ in the $\lim t \rightarrow \infty$ is a diffeomorphism and so its degree is 1. This shows that the degree of T_0 is also 1, and completes the proof that u_0 is globally c -convex. \square

REFERENCES

- [Br] Y. BRENIER: Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44** (1991), 375-417.
- [Ca1] L. A. CAFFARELLI: The regularity of mapping with a convex potential. *J. Amer. Math. Soc.* **5** (1992), 99-104.
- [Ca2] L. A. CAFFARELLI: Boundary regularity of maps with convex potentials II. *Ann. of Math.* **144** (1996), 453-496.
- [CNS] L. A. CAFFARELLI, L. NIRENBERG, AND J. SPRUCK: The Dirichlet problem for nonlinear second-order elliptic equations. I: Monge-Ampère equation, *Comm. Pure Appl. Math.* **37** (1984) 369-402.
- [Cb] E. CALABI: Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Michigan Math. J.* **5** (1958), 105-126.
- [Co] D. CORDERO-ERAUSQUIN: Sur le transport de mesures périodiques. *C. R. Acad. Sci. Paris Sér. I Math.* **329** (1999), 199-202.
- [D] P. DELANOË: Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator. *Ann. Inst. Henri Poincaré-Anal. Non Lin.* **8** (1991), 443-457.
- [DG] P. DELANOË AND Y. GE: Regularity of optimal transportation maps on compact, locally nearly spherical, manifolds. To appear in *J. Reine Angew. Math.*
- [DL] PH. DELANOË AND G. LOEPER: Gradient estimates for potentials of invertible gradient-mappings on the sphere. *Calc. Var. Partial Differential Equations* **26** (2006), no. 3, 297-311.
- [FKM1] A. FIGALLI, Y.-H. KIM AND R.J. MCCANN: Continuity and injectivity of optimal maps for non-negatively cross-curved costs. Preprint at arXiv:0911.3952.
- [FKM2] A. FIGALLI, Y.-H. KIM AND R.J. MCCANN: When is multidimensional screening a convex program? Preprint at arXiv:0912.3033.

- [FKM3] A.FIGALLI, Y.-H. KIM AND R.J. MCCANN: Regularity of optimal transport maps on multiple products of spheres. Preprint at arXiv:1006.1957.
- [FR] A. FIGALLI AND L. RIFFORD: Continuity of optimal transport maps on small deformations of S^2 . *Comm. Pure Appl. Math.*, **62** (2009), no. 12, 1670-1706.
- [FRV1] A.FIGALLI, L. RIFFORD AND C. VILLANI: Nearly round spheres look convex. Preprint 2009.
- [FRV] A. FIGALLI, L. RIFFORD AND C. VILLANI: Necessary and sufficient conditions for continuity of optimal transport maps on Riemannian manifolds. Preprint 2010.
- [GM] W. GANGBO AND R.J. MCCANN: The geometry of optimal transportation. *Acta Math.* **177** (1996), 113-161.
- [Ha] R. HAMILTON: Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), no. 2, 255-306.
- [Km] Y.-H. KIM: Counterexamples to continuity of optimal transportation on positively curved Riemannian manifolds. *Int. Math. Res. Not. IMRN* (2008) Art. ID rnn120, 15 pp.
- [KM1] Y.-H. KIM AND R.J. MCCANN: Continuity, curvature, and the general covariance of optimal transportation. *J. Eur. Math. Soc. (JEMS)* **12** (2010), 1009-1040.
- [KM2] Y.-H. KIM AND R.J. MCCANN: Towards the smoothness of optimal maps on Riemannian submersions and Riemannian products (of round spheres in particular). Preprint at arXiv: 0806.0351. To appear in *J. Reine Angew. Math.*
- [KMW] Y.H.KIM, R.J.MCCANN & M. WARREN: Pseudo-Riemannian geometry calibrates optimal transportation. Preprint 2009. To appear in *Math. Res. Lett.*
- [Kt] J. KITAGAWA: A parabolic flow toward solutions of the optimal transportation problem on domains with boundary. Preprint at arXiv:1007.4526.
- [Lb] G. LIEBERMANN: Second Order Parabolic Differential Equations 1996. World Scientific.
- [LY] P. LI AND S.T. YAU: On the parabolic kernel of the Schrödinger operator. *Acta Math.* **156** (1986), no. 3-4, 153-201.
- [LTW] J. LIU, N. TRUDINGER AND X.-J. WANG: Interior $C^{2,\alpha}$ -regularity for potential functions in optimal transportation. *Comm. Partial Differential Equations* **35** (2010) no. 1, 165-184.
- [Lo1] G. LOEPER: On the regularity of solutions of optimal transportation problems. *Acta Math.* **202** (2009), 241-283.
- [Lo2] G. LOEPER: Regularity of optimal maps on the sphere: The quadratic cost and the reflector antenna. To appear in *Arch. Ration. Mech. Anal.*
- [LV] G. LOEPER AND C. VILLANI: Regularity of optimal transport in curved geometry: the nonfocal case. *Duke Math. J.* **151** no. 3 (2010), 431-485.
- [Mc] R.J. MCCANN: Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* **11** (2001) 589-608.
- [MTW] X.N.MA, N.TRUDINGER & X.J.WANG: Regularity of potential functions of the optimal transport problem. *Arch. Ration. Mech. Anal.*, **177** (2005), no. 2, 151-183.
- [T] N. S. TRUDINGER: Recent developments in elliptic partial differential equations of Monge-Ampère type. In *International Congress of Mathematicians*, volume 3, pages 291-301. Eur. Math. Soc., Zürich, 2006.
- [TW1] N.S. TRUDINGER AND X.-J. WANG: On the second boundary value problem for Monge-Ampère type equations and optimal transportation. *Ann. Scuola Norm. Sup. Pisa*, **8** (2009) 1-32.
- [TW2] N.S. TRUDINGER AND X.-J. WANG: On Strict Convexity and Continuous Differentiability of Potential Functions in Optimal Transportation. *Arch. Ration. Mech. Anal.* **192** (2009), no. 3, 403-418.
- [TW3] N.S. TRUDINGER AND X.-J. WANG: On convexity notions in optimal transportation. Preprint.
- [Sch] O. SCHNÜRER: Flows toward reflectors. *Analysis (Munich)* **23** (2003) 261-275.
- [Wg] X.-J. WANG: On the design of a reflector antenna. *Inverse Problems* **12** (1996) 351-375.
- [U] J. URBAS: On the second boundary value problem for equations of Monge-Ampère type. *J. Reine Angew. Math.* **487** (1997), 115-124.

- [Wr] M. WARREN: A McLean Theorem for the moduli space of Lie solutions to Riemmanian transport equations. Preprint at arXiv:1006.1334v2.
- [V] C. VILLANI: Optimal transport: old and new. *Grundlehren der mathematischen Wissenschaften*. vol. 338. Springer 2009.

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